19. Convolution of Signals: Integral

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$$x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t - \tau)\, d\tau$$

Convolution is an integral that occurs frequently in engineering systems. The convolution of two signals $x(t)$ and $y(t)$ is denoted by $x(t) * y(t)$ or $x(t) \circledast y(t)$ or $(x * y)(\tau)$ and is defined as

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

Note: Almost all textbooks define convolution as

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t - \tau)\, d\tau$$

which is **exactly** the same definition. Just replace every variable t with τ and every τ with t, which is a simple change of variables. My definition is *more intuitive* (τ is time shift) and is consistent with correlation.

The three common techniques to calculate the convolution integral:

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

1. Substitute the signals into the mathematical equation and perform normal integration and mathematical manipulation.

2. Perform a graphical solution (flip, shift, multiply, find area, repeat).

3. Memorize the answer (after solving the integral once).

Q1. For the signals $x(t) = e^{-3t}u(t)$ and $y(t) = e^{-t}u(t)$, determine the convolution result $x(t) * y(t)$.

Q1. Solution. Substituting in the integral definition of convolution

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-\infty}^{\infty} e^{-3t}u(t)\, e^{-(\tau-t)}u(\tau - t)\, dt$$

Notice that $u(t) = 0$ for $t \in (-\infty, 0)$ and $u(\tau - t) = u\big(-(t - \tau)\big) = 0$ for $t \in (\tau, \infty)$. First, consider the case $\tau \geq 0$. Hence,

$$z(\tau) = \int_{0}^{\tau} e^{-3t}\, e^{-(\tau-t)}\, dt$$

$$z(\tau) = \int_0^\tau e^{-3t}\, e^{-(\tau - t)}\, dt$$

$$z(\tau) = \int_0^\tau e^{-3t}\, e^{-\tau}\, e^{t}\, dt = e^{-\tau} \int_0^\tau e^{-2t}\, dt = e^{-\tau} \left[\frac{e^{-2t}}{-2}\right]_0^\tau$$

$$z(\tau) = e^{-\tau} \times \frac{e^{-2\tau} - e^0}{-2} = e^{-\tau} \times \frac{1 - e^{-2\tau}}{2}$$

$$z(\tau) = \frac{e^{-\tau}}{2} - \frac{e^{-3\tau}}{2} = 0.5(e^{-\tau} - e^{-3\tau}), \quad \tau \geq 0$$

The second case is when $\tau < 0$?

Notice that $u(t) = 0$ for $t \in (-\infty, 0)$ and $u(\tau - t) = u\big(-(t - \tau)\big) = 0$ for $t \in (\tau, \infty)$.

So, for $\tau < 0$, the multiplication becomes

$$e^{-3t} u(t)\, e^{-(\tau - t)} u(\tau - t) = 0$$

And,

$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0, \quad \tau < 0$$

Hence, the full solution is,

$$z(\tau) = 0.5(e^{-\tau} - e^{-3\tau}) u(\tau)$$

Q2. For the signals $x(t) = 5e^{-2t}u(t)$ and $y(t) = 6\,u(t)$, determine the convolution result $x(t) * y(t)$.

Q2. Solution. Use the integral definition of convolution to get

$$z(\tau) = \begin{cases} 15(1 - e^{-2\tau}), & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}$$

Similarly

$$z(\tau) = 15(1 - e^{-2\tau})u(\tau)$$